

Gapless inhomogeneous superfluid phase with spin-dependent disorder

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We show that the presence of a spin-dependent random potential in a superconductor or a superfluid atomic gas leads to distinct transitions at which the energy gap and average order parameter vanish, generating an intermediate gapless superfluid phase, in marked contrast to the case of spin-symmetric randomness where no such gapless superfluid phase is seen. By allowing the pairing amplitude to become inhomogeneous, the gapless superconducting phase persists to considerably higher disorder compared with the prediction of Abrikosov-Gorkov. The low-lying excited states are located predominantly in regions where the pairing amplitude vanishes and coexist with the superfluid regions with a finite pairing. Our results are based on inhomogeneous Bogoliubov-de Gennes mean field theory for a two dimensional attractive Hubbard model with spin-dependent disorder.

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Introduction: The interplay of disorder and strong interactions and the mechanisms underlying metal-insulator transitions is still an open and very important question. Traditionally, this problem has been discussed in the context of solid state systems. More recently it has become possible to emulate quantum Hamiltonians using ultracold atomic gases [1], opening up a new dialog between condensed matter and atomic physics leading to new insights into the effects of correlation on randomness.

The Anderson model for localization [2], first proposed in 1958, describes the possibility of localized electronic states formed by quantum interference of waves in a random potential. Later, in the famous “gang of 4” paper, it was shown using renormalization group flows of the conductance that a system in two or lower dimensions would always flow to a localized state for arbitrarily weak disorder. In contrast, in three dimensions there is a critical disorder separating localized and conducting states [3]. Over the years the precise verification of the Anderson model has not been possible because in real systems interactions cannot be ignored, and, moreover, become increasingly important as the states get more localized near the metal-insulator transition. However, very recently in ultracold atomic gases the Anderson model has been emulated for the first time by tuning the interactions to zero [4] using Feshbach resonance and the localized wave function has been visualized. In these experiments, randomness is introduced through optical speckle [5] or bichromatic lattices [6].

If attractive interactions between fermions are turned on, such systems become superconducting (for charged electrons) or superfluid (for neutral fermions). These states can exist in two dimensions in a disordered system below a critical disorder even when all the single particle states are localized. The ensuing superconductor-insulator transition has been studied in considerable de-

tail in the literature [7–9]. A particularly intriguing aspect of cold atoms is the ability to tune the interactions either through the Feshbach resonance or through optical lattices. In the limit of strong attractive interactions between fermions the problem crosses over to bosons in a random potential [10–14] where new glassy phases are expected for large disorder. In this way, the complex interplay of interactions and disorder is combined with BCS to BEC crossover physics [15].

In this paper, we investigate the effect of a spin-dependent random potential on a superconductor. A related problem, that of magnetic impurities in a superconductor, was studied by Abrikosov and Gorkov [16] who found that the superfluid density was suppressed for fairly small impurity concentrations. These diagrammatic calculations were performed at weak coupling in the BCS regime. Here, in contrast, we investigate the intermediate to strongly coupled BCS-BEC crossover regime coupled with magnetic disorder, a problem that has not been explored so far, and which is now of very direct experimental relevance to atomic gases. We show that within Bogoliubov-de Gennes mean field theory spin-dependent disorder exhibits qualitatively different features from the conventional symmetric case where both species feel the random potential. In particular, the gap and order parameter are driven to zero with increasing disorder, whereas they remain nonzero, and can even increase, when both species see the same energy landscape. This destruction of pairing has also been seen in BdG studies in the presence of spin-independent order and a uniform Zeeman field [9].

Spin dependent lattices can be achieved, for instance, by utilizing two atomic hyperfine transitions, D1 and D2 (ground to excited states) of the fermionic alkali atom ^{40}K , which can be individually excited by right- and left-circular polarized light respectively (according

to selection rules). By creating a lattice with independently tunable components of right- and left-circular polarized standing-wave light, a spin-dependent lattice is realized [17]. The two fermionic “spin” species in this case are the two hyperfine states of the degenerate ground-state manifold split by a small uniform magnetic field. On the other hand, disorder can be superimposed on the periodic optical lattices created by interfering counter-propagating lasers by means fine-grained speckle fields [4, 5]. Accordingly, in principle, spin-dependent disorder can be realized by superimposing on the lattice separate laser beams that have been passed through a diffuser and focused at the lattice location. A suitable diffuser would be able to modulate the lattice intensity at each lattice site in a random way, with the randomness uncorrelated on the scale of lattice constants [18, 19].

Model and Methodology: Our starting point is the attractive fermion Hubbard Hamiltonian with spin dependent disorder,

$$H = -t \sum_{\langle \mathbf{ij} \rangle, \sigma} (c_{\mathbf{j}\sigma}^\dagger c_{\mathbf{i}\sigma} + c_{\mathbf{i}\sigma}^\dagger c_{\mathbf{j}\sigma}) \quad (1)$$

$$- |U| \sum_{\mathbf{i}} n_{\mathbf{i}\uparrow} n_{\mathbf{i}\downarrow} + \sum_{\mathbf{i}\sigma} (\epsilon_{\mathbf{i}\sigma} - \mu_\sigma) n_{\mathbf{i}\sigma}$$

Here $c_{\mathbf{i}\sigma}^\dagger$ ($c_{\mathbf{i}\sigma}$) are fermion creation(destruction) operators at site \mathbf{i} for fermionic species (spin or hyperfine level) $\sigma = \uparrow, \downarrow$. t is the hopping amplitude between near neighbor sites $\langle \mathbf{ij} \rangle$ on a square lattice. $t = 1$ is the energy unit. $-|U|$ is an on-site attraction, and μ_σ is the chemical potential, which in general will depend on the spin index σ in order to maintain equal populations when the randomness is different for the two species. $\epsilon_{\mathbf{i}\sigma}$ is a local site energy which, in the conventional “Anderson-Hubbard” Hamiltonian, is chosen to be independent of fermion species σ . Here we focus on the situation where $\epsilon_{\mathbf{i}\downarrow} = 0$ and $\epsilon_{\mathbf{i}\uparrow}$ is uniformly distributed on $[-V, V]$.

In the clean limit, the square lattice attractive Hubbard model is a supersolid (simultaneous long-range charge density wave and s -wave pair correlations) at $T = 0$ and density $\rho = 1$ (half-filling). When doped to $\rho \neq 1$, the degeneracy is broken, and the attractive Hubbard Hamiltonian exhibits a finite temperature Berezinskii-Kosterlitz-Thouless transition to a superconducting phase with T_c on the order of $0.1t$ [20]. In the limit of strong attraction, this model crosses over to the Bose-Hubbard model of tightly bound pairs[21].

Anderson’s theorem provides a first insight into the effect of randomness on the clean attractive Hubbard model, suggesting that, at least for weak disorder, superconductivity will survive. A number of numerical studies have quantitatively explored this model [22]. Here we will focus our comparison on the Bogoliubov-de Gennes (BdG) treatment of Ghosal *et al.*[7] since that is also the methodology employed here.

The interaction term in H can be decoupled in different

(charge, pairing, spin) channels. Since $U < 0$ we focus on pairing and write,

$$H_{\text{eff}} = -t \sum_{\langle \mathbf{ij} \rangle, \sigma} (c_{\mathbf{i}\sigma}^\dagger c_{\mathbf{j}\sigma} + c_{\mathbf{j}\sigma}^\dagger c_{\mathbf{i}\sigma}) + \sum_{\mathbf{i}\sigma} (\epsilon_{\mathbf{i}\sigma} - \tilde{\mu}_{\mathbf{i}\sigma}) n_{\mathbf{i}\sigma}$$

$$+ \sum_{\mathbf{i}} [\Delta_{\mathbf{i}} c_{\mathbf{i}\uparrow}^\dagger c_{\mathbf{i}\downarrow}^\dagger + \Delta_{\mathbf{i}}^* c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow}] \quad (2)$$

where $\tilde{\mu}_{\mathbf{i}\sigma} = \mu_\sigma + |U| \langle n_{\mathbf{i}, -\sigma} \rangle$. H_{eff} is diagonalized via the Bogoliubov transformation,

$$c_{\mathbf{i}\uparrow} = \sum_n [\gamma_{n\uparrow} u_{\mathbf{i}n} - \gamma_{n\downarrow}^\dagger v_{\mathbf{i}n}^*]$$

$$c_{\mathbf{i}\downarrow} = \sum_n [\gamma_{n\downarrow} u_{\mathbf{i}n} + \gamma_{n\uparrow}^\dagger v_{\mathbf{i}n}^*] \quad (3)$$

In the clean system the eigenfunctions $u_{\mathbf{i}n}$ and $v_{\mathbf{i}n}$ are plane waves. In the presence of disorder they are obtained by numerical diagonalization. The local order parameter and density are determined self-consistently,

$$\Delta_{\mathbf{i}} = -|U| \langle c_{\mathbf{i}\downarrow} c_{\mathbf{i}\uparrow} \rangle = -|U| \sum_n f(E_n) u_{\mathbf{i}n} v_{\mathbf{i}n}^* \quad (4)$$

$$\langle n_{\mathbf{i}\uparrow} \rangle = \sum_n f(E_n) |u_{\mathbf{i}n}|^2 \quad \langle n_{\mathbf{i}\downarrow} \rangle = \sum_n f(-E_n) |v_{\mathbf{i}n}|^2$$

(where f is the Fermi function), as are the chemical potentials required to achieve the desired density[7]. These self-consistency conditions are equivalent to minimizing the free energy. The correctness and efficiency of our codes have been checked by using different initial configurations of particle density and local pairing amplitude and checking for convergence. As expected, we found more iterations are needed for convergence near phase transitions. We define a spatially averaged order parameter, Δ_{op} , from $\Delta_{\mathbf{i}}$. The BdG spectrum can be used to determine the energy gap, E_{gap} , which is the lowest eigenvalue above the chemical potential. The spectrum and eigenfunctions also determine the density of states. Note that unlike the spin-independent case the eigenvalues do not come in \pm pairs and the distances of closest eigenvalues below and above the chemical potential are in general different.

The key conclusions of the BdG treatment of Ghosal *et al.* [7] for the usual spin-symmetric case $\epsilon_{\mathbf{i}\uparrow} = \epsilon_{\mathbf{i}\downarrow}$ are as follows: As disorder is increased the energy gap and lattice-averaged order parameter decrease, but never go to zero. In particular, even though the distribution of local pairing amplitudes has significant weight near zero, a finite spectral gap persists owing to remnant superconducting islands and, importantly, significant overlap of the low energy excited states with these islands. Anderson’s theorem concerning the survival of superconductivity is obtained under two assumptions: pairing of exact eigenstates and a further assumption that the kernel in the gap equation is spatially uniform. It is possible to generalize the calculation within pairing of exact eigenstates to allow for spatial structure in the pairing amplitude. BdG calculations go beyond pairing of exact

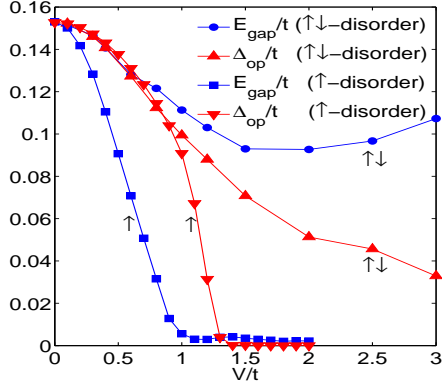


FIG. 1. The lattice-averaged order parameter Δ_{op} and the energy gap E_{gap} are shown as functions of the disorder strength V . In the spin-symmetric case considered in [7], these quantities never go to zero even at large V . In contrast, when the disorder is applied only to one spin species, sharp transitions are observed. The small “tail” in E_{gap} goes to zero as the lattice size increases. $N = 24 \times 24$, $\langle n \rangle = 0.875$, $U/t = -1.5$.

eigenstates and allow for a full treatment of the amplitude fluctuations in response to an underlying random potential. In order to see either the finite temperature phase transition from a superfluid to a non-superfluid state or the quantum phase transition from a superfluid to an insulator, thermal and quantum phase fluctuations must be included in the inhomogeneous BdG state.

We will see that even at the level of pairing of exact eigenstates or BdG the situation is dramatically transformed by spin-dependent disorder.

Results: Figure 1 shows the evolution of the energy gap E_{gap} and order parameter Δ_{op} with increasing randomness V for density $\langle n \rangle = 0.875$ and on-site attraction $U = -1.5t$. In the spin-symmetric case, confirming the results of [7], we find E_{gap} and Δ_{op} do not vanish. However, in the spin-asymmetric case, when $\epsilon_{i\downarrow} = 0$, there are instead sharp transitions for both quantities. Interestingly, E_{gap} and Δ_{op} do not vanish simultaneously, but instead three phases are present: a superconductor characterized by E_{gap} and Δ_{op} both nonzero at small disorder V , a gapless superconductor [16, 23, 24] at intermediate V , and a $E_{\text{gap}} = \Delta_{\text{op}} = 0$ phase at large V .

The gapless superfluid has also been observed in other situations in which spin symmetry is broken, *e.g.* the mismatched Fermi surface considered in [25]. In such situations, paired regions of the Fermi sea coexist with regions occupied by only a single spin species. These unpaired pockets lead to a gapless superfluid phase between the normal and BCS regimes, much as occurs in Fig. 1. Indeed, as we shall demonstrate later, in the attractive Anderson Hubbard Hamiltonian considered here, the real space distribution of Δ_i , indicates coexisting *real space* domains which are in direct analogy to the coexisting *momentum space* domains of [25] and related work.

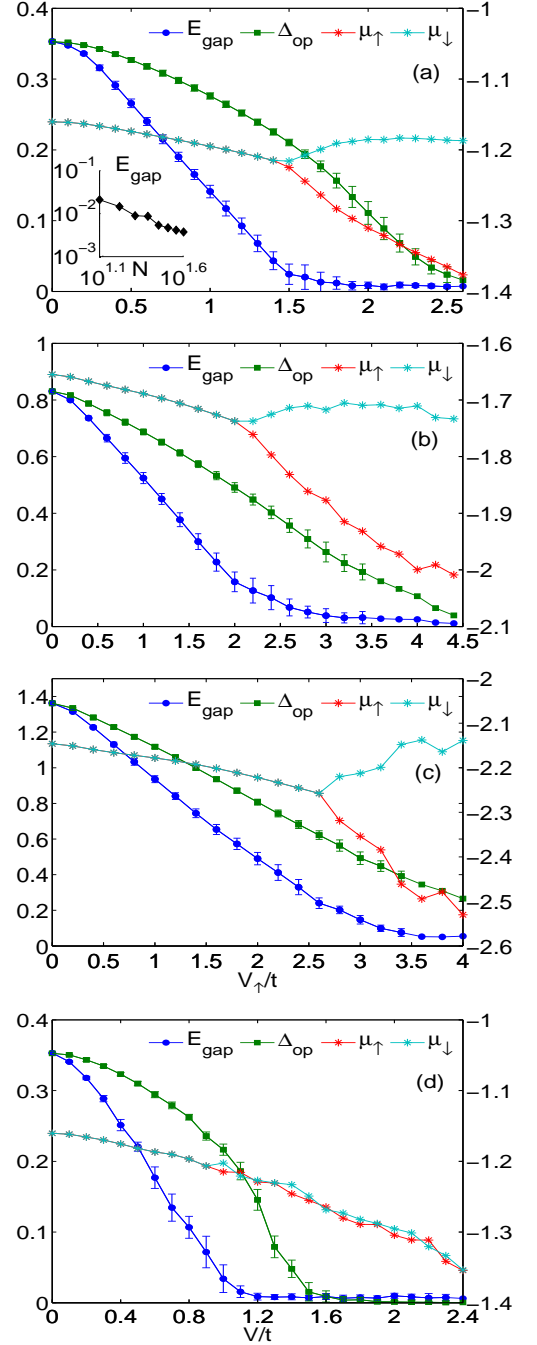


FIG. 2. The energy gap E_{gap} , average order parameter Δ_{op} (left y -axis) and chemical potentials μ_{σ} (right y -axis) are shown as functions of disorder strength V . Panels (a,b,c) show results for the case when $-V < \epsilon_{i\uparrow} < V$ and $\epsilon_{i\downarrow} = 0$ for three values, $U/t = -2, -3, -4$. Panel (d) shows a case when both $\epsilon_{i\sigma}$ are uniformly distributed on $[-V, V]$ but are chosen independently, for $U/t = -2$. The qualitative behavior for these two types of spin-asymmetric disorder is similar: There are two transitions, first from gapped to gapless superfluid and then to a phase in which $\Delta_{\text{op}} = 0$ as well. As expected, μ_{σ} is spin-independent in the situation of panel (d) when both species see disorder of the same overall strength (although distinct realizations). $N = 24 \times 24$, $\langle n \rangle = 0.875$. The results are averaged over 10 realizations of randomness.

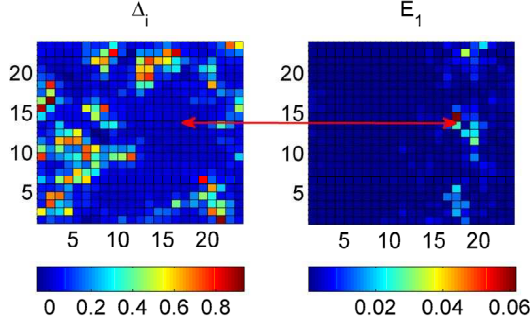


FIG. 3. Left panel: The distribution of the local order parameter Δ_i . Right panel: The first excited state wave function. Here $U = -3t$ and $V_\uparrow = 4t$. The average order parameter is non-zero, but because the first excited state has weight in the region where the local values are zero there is no energy gap. Hence the system is a gapless superfluid.

Figures 2(abc) provide further details for the evolution of E_{gap} and Δ_{op} for three interaction strengths, $U/t = -2.0, -3.0$ and -4.0 . The transition points move to larger disorder strength, as expected, as the attractive interaction is increased. At all U/t , there is a residual nonzero value of E_{gap} which scales to zero with increasing system size. In fact, the transition point from gapped to gapless superconductor is best signalled by a bifurcation of the chemical potentials $\mu_\uparrow \neq \mu_\downarrow$ required to maintain a spin-balanced population $\rho_\uparrow = \rho_\downarrow$.

In order to determine whether such transitions are generic to the breaking of spin symmetry in the disorder we consider, we show in Fig. 2d a case where both species see disorder drawn from the same distribution $[-V, V]$ but with different realizations. That is, spin \uparrow and spin \downarrow fermions see different disordered landscapes. The basic features are the same as the case when only one species sees the disorder (Figs. 2abc): there are two separate transitions at which $E_{\text{gap}} \rightarrow 0$ and $\Delta_{\text{op}} \rightarrow 0$. As expected, the chemical potentials do not bifurcate in the case of Fig. 2d. Nevertheless, there is still a signature of the vanishing of E_{gap} in the scatter of the chemical potential data.

To understand the physics of the gapless superfluid we show, in Fig. 3, the spatial distribution of the local order parameter Δ_i (left) and the first excited state wave function Ψ_i^* (right). The system is highly inhomogeneous and, although many sites have nonzero Δ_i , so that the average order parameter is substantial, there are also large domains where the local order parameter vanishes. The first excited state lives predominantly on the latter set of sites, leading to a vanishing energy gap. In the gapped superfluid phase at weaker disorder, Ψ^* substantially overlaps regions of non-zero order parameter. Similarly, in the canonical case of spin-independent disorder the sites at which the first excited state has non-zero amplitude *coincide* with those of non-zero order parameter.

We can also infer the phase from the density of states

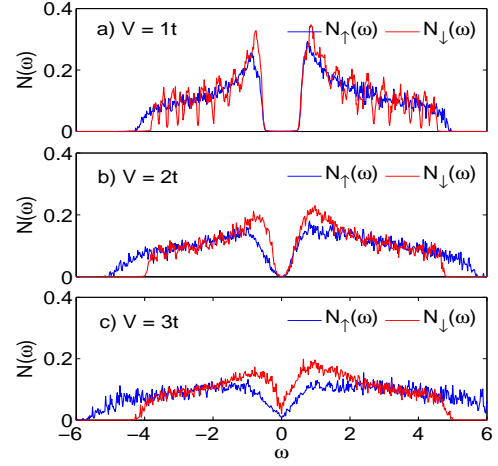


FIG. 4. Density of states at $U = -3t$ for (top to bottom) $V_\uparrow = 1.0t, 2.0t$ and $3.0t$. The gap closes for $V_\uparrow = 3.0t$. See also Fig. 2(b).

$N(\omega)$ obtained as a histogram of the BdG eigenvalues. Note again that the breaking of spin symmetry destroys the usual appearance of eigenvalues in $\pm E_n$ pairs. Figure 4 shows the density of states

$$N_\uparrow(\omega) = \frac{1}{N} \sum_{n,i} |u_{in}|^2 \delta(\omega - E_n)$$

$$N_\downarrow(\omega) = \frac{1}{N} \sum_{n,i} |v_{in}|^2 \delta(\omega + E_n) \quad (5)$$

for $U = -3t$ and a range of disorder strengths. The gap is nonzero for $V_\uparrow = 1.0t$ but has closed by the time $V_\uparrow = 3.0t$, in agreement with Fig. 2(b).

Discussion: The effects of spin-dependent disorder on the superconducting phase in the attractive Hubbard Hamiltonian differ fundamentally from conventional chemical potential disorder. Sharp transitions at which the energy gap and average order parameter vanish can clearly be seen as the randomness increases. This can be attributed at the fundamental level to the breaking of time reversal symmetry, which can arise in the solid state by the spin-dependent scattering produced by magnetic impurities. In fact, the spin-dependent site energy $\epsilon_{i\uparrow}n_{i\uparrow} + \epsilon_{i\downarrow}n_{i\downarrow}$ in Eq. 2, can be thought of as a combination of a local chemical potential $\frac{1}{2}(\epsilon_{i\uparrow} + \epsilon_{i\downarrow})(n_{i\uparrow} + n_{i\downarrow})$ and a local Zeeman field, $\frac{1}{2}(\epsilon_{i\uparrow} - \epsilon_{i\downarrow})(n_{i\uparrow} - n_{i\downarrow})$. The Zeeman term breaks time reversal symmetry, providing one way to interpret the results presented here. The occurrence of sharp transitions can also be understood in analogy with exotic paired states in other situations like Fulde-Ferrell-Larkin-Ovchinnikov[26, 27] with mismatched Fermi surfaces[25]. In these systems, electrons of one species with a particular *momentum* cannot find partners, and hence a superfluid and unpaired electrons coexist. An analogous phenomenon occurs in *real space* in the Hamiltonian consid-

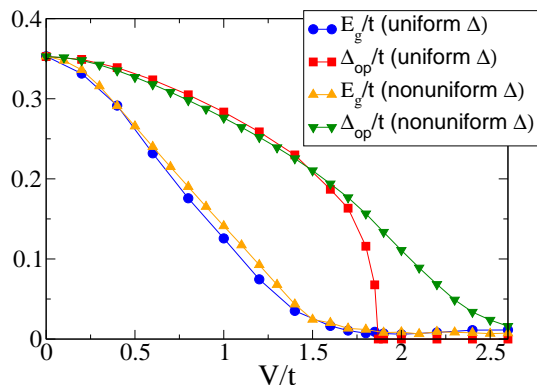


FIG. 5. The behavior of the energy gap and the order parameter for the same disorder strength for the case when the local order parameter is forced to be uniform to conform to the conditions for which the classic Abrikosov-Gorkov theory [16] was developed and its comparison with a spatially non-uniform pairing amplitude. Notice that a finite order parameter can exist up to much higher disorder when the pairing amplitude is allowed to be non-uniform. $N = 24 \times 24$, $U/t = -2$, $\langle n \rangle = 0.875$.

ered in this paper: the spin-dependent randomness provided regions of the lattice where fermions cannot find partners, while in other regions pairing can continue.

Comparison with Abrikosov-Gorkov theory: It is useful to compare our results in Figs.2 and 3 against the early work by Abrikosov and Gorkov (AG) [16] where they first predicted the possibility of gapless superconductivity for spin-dependent scattering, a phase with finite order parameter but no gap. Their result was obtained within perturbation theory in a regime of weak disorder assuming that the local pairing amplitude is homogeneous.

In Fig.5 we compare our results for the energy gap and order parameter obtained from the inhomogeneous BdG equations as discussed above against a restricted calculation where we enforce a spatially uniform local pairing amplitude $\bar{\Delta} \equiv (1/N) \sum_i \Delta(\mathbf{r}_i)$ by finding a self-consistent solution for the average pairing amplitude $\bar{\Delta}$. The significant aspects of these results are: (1) When the local pairing amplitude is forced to be uniform, there is a small region in disorder strength where a gapless phase exists, in agreement with AG theory [16]. (2) When the local pairing amplitude is allowed to vary spatially, a much larger region of gapless superconductivity opens up. The gap is not affected much by the local pairing amplitude variations but the disorder strength up to which a finite order parameter exists as deduced from long range superconducting correlations is greatly enhanced.

In that respect the gapless superconducting phase that is reported here is different in nature from that in the earlier work because inhomogeneity of the local order parameter plays a vital role in enhancing the order parameter.

Conclusion: A key goal of optical lattice emulation is to

observe Quantum Phase Transitions (QPTs) in ultra-cold atomic gases, analogous to those believed to underlie condensed matter phenomena such as high temperature superconductivity. Our work suggests that, for exploring QPTs associated with the interplay of attractive interactions and disorder, the sorts of spin-dependent lattices achieved experimentally [18, 19, 25] might be especially useful.

Several fundamental conceptual questions remain to be addressed by these experiments, and by more precise theoretical approaches such as Quantum Monte Carlo. One is the construction of a unified phase diagram in which spin-dependence arising from the scattering, spin dependent disorder considered here, and from imbalanced populations, FFLO physics, are both included. Specific issues include the conditions for the existence of gapless superfluidity and whether an inhomogeneous order parameter is essential for its formation. Another outstanding question concerns transport properties: Starting from the weakly interacting limit, in which one species feels disorder and the other does not, the system is ‘half-metal’ with distinct localized and itinerant components. As an attractive interaction U is turned on, the delocalized particles will feel an *induced* randomness as they interact with the localized species. Is there a critical U separating distinct half-metallic phases from phases in which the two components have identical transport characteristics?

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